B.Sc. IIIrd Sem

Paper – X Differential Equations (Mathematics)

Model Answer / Suggestive Answers

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1. (i). order – 4

(ii). Bernoulli's equation – $\frac{dy}{dx} + Py = Q \cdot y^n$
where P and Q are function of x.

(iii). $a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = Q$

where $a_0, a_1, a_2, \ldots, a_n$ are constant and Q is a function of x or constant.

(iv). $CF = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

$= c_1 e^{\alpha x} \cos(\beta x + c_2)$

$= c_1 e^{\alpha x} \sin(\beta x + c_2)$

(v). Homogeneous linear equation

$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = Q$

(vi). If $y'' + Py' + Qy = R$ Then P.I. $v = e^{-\frac{1}{2}\int P dx}$
or $u = e^{-\frac{1}{2}\int P dx}$

(vii). (a) $Q'_{n+1} - Q'_{n-1} = (2n+1) Q_n$.

(b) $n Q'_{n+1} + (n+1) Q'_{n-1} = (2n+1) x Q'_n$

(c) $(2n+1) x Q_n = (n+1) Q_{n+1} + n Q_{n-1}$.

(d) $(2n+1)(1-x^2) Q'_n = n(n+1)(Q_{n-1} - Q_{n+1})$.

(viii). $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = \frac{2}{\pi x}$.

(2.) $\dfrac{dy}{dx} = \dfrac{x^2+3y^2}{3x^2+y^2}$

Put $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{x^2+3v^2x^2}{3x^2+v^2x^2} = \frac{1+3v^2}{3+v^2}$$

$$x\frac{dv}{dx} = \frac{1+3v^2}{3+v^2} - v = \frac{1+3v^2-3v-v^3}{3+v^2}$$

$$\int\frac{dx}{x} = \int\frac{3+v^2}{(1-v)^3}dv = \int\left(\frac{1}{1-v} + \frac{-2}{(1-v)^2} + \frac{4}{(1-v)^3}\right)dv$$

$$= -\log(1-v) - \frac{2.(1-v)^{-1}}{(-1)(-1)} + \frac{4(1-v)^{-2}}{-2\times-1} + c$$

$$\log x.(1-v) = \frac{4}{(1-v)^2} - \frac{2}{1-v} + c$$

$$\log(x-y) = \frac{4x^2}{(x-y)^2} - \frac{2x}{(x-y)} + c = \frac{4x^2 - 2x(x-y)}{(x-y)^2} + c$$

$$= \frac{4x^2 - 2x^2 + 2xy}{(x-y)^2} + c$$

$$\log(x-y) = \frac{2xy + 2x^2}{(x-y)^2} + c$$

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(3). (a) $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$

A.E. $m^3 - 6m^2 + 11m - 6 = 0$

$m^3 - m^2 - 5m^2 + 5m + 6m - 6 = 0$

$(m-1)(m^2 - 5m + 6) = 0$

$(m-1)(m-3)(m-2) = 0$

$m = 1, 3, +2$

$y = cf = c_1 e^{x} + c_2 e^{3x} + c_3 e^{2x}.$

(b). $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 4y = 0$

A.E. $m^2 + 5m + 4 = 0$

$(m+1)(m+4) = 0$

$m = -1, -4$

$y = cf = c_1 e^{-x} + c_2 e^{-4x}$

4. $(D^2-4D+4)y = e^{2x}+x^3+\cos 2x.$

A.E. $m^2-4m+4=0$

$(m-2)^2 = 0 \Rightarrow m = 2, 2$

$CF = (c_1+c_2x)\,e^{2x}$

$$P.I. = \frac{1}{D^2-4D+4}\left(e^{2x}+x^3+\cos 2x\right)$$

$$= \frac{1}{2^2-4\times 2+4}\,e^{2x} + \frac{1}{4}\left[1+\frac{D^2-4D}{4}\right]^{-1}\cdot x^3 + \frac{1}{-2^2-4D+4}\cos 2x$$

$\because f(2) = 0$

$$\therefore P.I. = x\cdot\frac{1}{2D-4}\,e^{2x} + \frac{1}{4}\left[1-\frac{D^2-4D}{4}+\left(\frac{D^2-4D}{4}\right)^2-\left(\frac{D^2-4D}{4}\right)^3+\dots\right]x^3 - \frac{1}{4D}\cos 2x$$

$$= x\cdot\frac{1}{2\times 2-4}\,e^{2x} + \frac{1}{4}\left[1-\frac{D^2}{4}+D+D^2-\frac{\cancel{8}^{\,2}D^3}{\cancel{4}}+D^3\right]x^3 - \frac{1}{4}\cdot\frac{\sin 2x}{2}$$

Again $f'(2) = 0$

$$\therefore P.I. = x^2\cdot\frac{1}{2}\,e^{2x} + \frac{1}{4}\left[1+D+\frac{3D^2}{4}-D^3\right]x^3 - \frac{1}{8}\sin 2x$$

$$= \frac{x^2e^{2x}}{2} + \frac{1}{4}\left(x^3+3x^2+\frac{\cancel{18}^{\,9}x}{\cancel{4}_{\,2}}-6\right) - \frac{1}{8}\sin 2x$$

$y = CF + P.I.$

$$= (c_1+c_2x)\,e^{2x} + \frac{1}{8}\left(2x^3+6x^2+9x-12\right) - \frac{1}{8}\sin 2x + \frac{x^2e^{2x}}{2}$$

5. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2\log x.$

Put $z = \log x$ or $x = e^z$

Then $x\dfrac{dy}{dx} = Dy$, $x^2\dfrac{d^2y}{dx^2} = D(D-1)y$

$\therefore D(D-1)y - Dy - 3y = e^{2z}\cdot z$

$(D^2-2D-3)y = z\cdot e^{2z}$

A.E. $m^2-2m-3 = 0 \Rightarrow$

$(m-3)(m+1) = 0 \Rightarrow m = -1, 3$

$CF = c_1e^{-z} + c_2e^{3z} = c_1 \ldots + c_2x^3$

(4) $P.I. = \frac{1}{D^2-2D-3} \cdot z \cdot e^{2z}$

$$= e^{2z} \frac{1}{(D+2)^2-2(D+2)-3} z$$

$$= e^{2z} \frac{1}{D^2+\cancel{4}+4D-2D-\cancel{4}-3} z$$

$$= e^{2z} \frac{1}{D^2+2D-3} z = \frac{e^{2z}}{-3}\left[1-\frac{D^2+2D}{3}\right]^{-1} \cdot z$$

$$= -\frac{e^{2z}}{3}\left[1+\frac{2D}{3}\right] z \quad = -\frac{e^{2z}}{3}\left(z+\frac{2}{3}\right)$$

$$= -\frac{x^2}{3}(\log x + 2/3)$$

$y = CF + P.I \quad = \frac{c_1}{x} + c_2 x^3 - \frac{x^2}{3}(\log x + 2/3)$

(6.) $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = -3e^{x^2}\sin 2x.$

Compairing with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

$P = -4x, \; Q = 4x^2-1, \; R = -3e^{x^2}\sin 2x$

In order to remove the first derivative,

$$v = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int -4x\,dx} = e^{x^2}$$

on putting $y = uv$, the normal form is

$$\frac{d^2u}{dx^2} + Q_1 u = R_1 \longrightarrow (1)$$

where $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$

$$= 1$$

$R_1 = R/v \quad = -3\sin 2x$

$\therefore$ eqn (1) becomes

$$\frac{d^2u}{dx^2} + u = -3\sin 2x.$$

$$(D^2+1)u = -3\sin 2x.$$

A.E. $m^2+1=0 \Rightarrow m = 0 \pm i$

$CF = c_1 \cos x + c_2 \sin x$

$$P.I. = \frac{1}{D^2+1}\cdot(-3\sin 2x)$$

$$= -\frac{3\sin 2x}{-2^2+1} = \sin 2x$$

$\therefore u = CF + P.I. = c_1\cos x + c_2 \sin x + \sin 2x$

$$\Rightarrow y = u\cdot v = (c_1\cos x + c_2\sin x + \sin 2x)\cdot e^{x^2}$$

(7) $$P_n(x) = \frac{1}{2^n\, n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n.$$

Proof: Let $v = (x^2-1)^n$

$$\Rightarrow \frac{dv}{dx} = n(x^2-1)^{n-1}\cdot 2x$$

$$\Rightarrow (x^2-1)\frac{dv}{dx} = 2nx\cdot v$$

diff. $(n+1)$ times by L.T.

$$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + {}^{n+1}C_1\cdot 2x\cdot\frac{d^{n+1}v}{dx^{n+1}} + {}^{n+1}C_2\cdot(2)\frac{d^n v}{dx^n}$$

$$= 2n\left[x\frac{d^{n+1}v}{dx^{n+1}} + {}^{n+1}C_1\frac{d^n v}{dx^n}\right]$$

$$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + 2x\left({}^{n+1}C_1 - n\right)\frac{d^{n+1}v}{dx^{n+1}} + 2\left({}^{n+1}C_2 - n\cdot{}^{n+1}C_1\right)\frac{d^n v}{dx^n} = 0$$

$$\Rightarrow (x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + 2x\frac{d^{n+1}v}{dx^{n+1}} - n(n+1)\frac{d^n v}{dx^n} = 0 \longrightarrow (1)$$

Put $\frac{d^n v}{dx^n} = u$ in (1)

$$(x^2-1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)y = 0$$

$$\text{or } (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

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(6) which shows that $y = \frac{d^n u}{dx^n}$ is a sol. of Legendre's equation.

$\therefore c \cdot \frac{d^n u}{dx^n} = P_n(x)$ where c is a constant $\longrightarrow$ (2)

But $u = (x^2-1)^n = (x+1)^n \cdot (x-1)^n$

so that $\frac{d^n u}{dx^n} = (x+1)^n \cdot \frac{d^n}{dx^n}(x-1)^n + {}^nC_1\, n(x+1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}(x-1)^n + \cdots$

$+ (x-1) \frac{d^n}{dx^n}(x+1)^n$

when $x = 1$, then $\frac{d^n u}{dx^n} = 2^n \cdot n! \longrightarrow$ (3)

Put $x=1$ in (2) and using (3)

$$\therefore c \cdot 2^n\, n! = P_n(1) = 1$$

$$\Rightarrow c = \frac{1}{2^n \cdot n!}$$

$\therefore$ (2) becomes $P_n(x) = \frac{1}{2^n\, n!} \frac{d^n u}{dx^n}$

$$= \frac{1}{2^n\, n!} \frac{d^n}{dx^n}(x^2-1)^n$$

(8). $$\frac{1+z}{z\sqrt{1-2xz+z^2}} - \frac{1}{z} = \sum_{n=0}^{\infty} (P_n + P_{n+1}) \cdot z^n.$$

RHS. $= \sum_{n=0}^{\infty} (P_n + P_{n+1}) z^n = \sum P_n z^n + \sum P_{n+1} z^n$

$= \sum P_n z^n + \frac{1}{z} \sum P_{n+1} \cdot z^{n+1} \longrightarrow$ (1)

But $\sum P_n z^n = P_0 + z P_1 + z^2 P_2 + \cdots$

and $\sum P_{n+1} z^{n+1} = z P_1 + z^2 P_2 + \cdots$

$= -P_0 + P_0 + z P_1 + z^2 P_2 + \cdots$

$= -P_0 + \sum z^n P_n$

$\therefore$ RHS. $= \sum z^n P_n + \frac{1}{z}(-P_0 + \sum z^n P_n)$

$= (1 + \frac{1}{z}) \sum z^n P_n - \frac{P_0}{z}$

$= \frac{1+z}{z \cdot \sqrt{1-2xz+z^2}} - \frac{1}{z}$ LHS.